# Two-loop $\beta$-function from the exact renormalization group

Peter Kopietz

*Institut für Theoretische Physik der Universität Göttingen, Bunsenstrasse 9, 37073 Göttingen, Germany* [1]

**Abstract**

We calculate the two-loop renormalization group (RG) $\beta$-function of a massless scalar field theory from the irreducible version of Polchinski's exact RG flow equation. To obtain the correct two-loop result within this method, it is necessary to take the full momentum-dependence of the irreducible four-point vertex and the six-point vertex into account. Although the same calculation within the orthodox field theory method is less tedious, the flow equation method makes no assumptions about the renormalizability of the theory, and promises to be useful for performing two-loop calculations for non-renormalizable condensed-matter systems. We pay particular attention to the problem of the field rescaling and the effect of the associated exponent $\eta$ on the RG flow.
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## 1 Introduction

There are two different ways of formulating renormalization group (RG) transformations: the first formulation [1], which we shall call the field theory method, relies on the fact that in a renormalizable field theory all physical quantities can be expressed in terms of a finite number of renormalized couplings which can be defined at an arbitrary scale $\mu$. Because the bare theory is defined without reference to $\mu$, any change in the renormalized correlation functions in response to a variation of $\mu$ must be compensated by a corresponding change in the renormalized couplings, such that the bare correlation functions remain independent of $\mu$. This relation is usually expressed in terms of a partial differential equation for the renormalized correlation functions,

[1] Address after September 2000: Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Strasse 10, 60054 Frankfurt am Main.

the Callan-Symanzik equation. The Gell-Mann-Low $\beta$-functions describe how the renormalized couplings must depend on $\mu$ to guarantee that for fixed bare couplings the bare theory is independent of $\mu$.

The other, more modern formulation of the RG was pioneered by Wilson [2]. In this approach, which has been very fruitful in statistical physics and in condensed matter theory [3], one starts from a local bare action $S\{\phi\}$ with some ultraviolet cutoff $\Lambda_0$, and integrates out the degrees of freedom for momenta down to some smaller cutoff $\Lambda < \Lambda_0$. After this elimination of the degrees of freedom, the momenta and in general also the fields have to be properly rescaled in order to obtain a fixed point of the RG [4]. In this way one obtains an effective action with a new ultraviolet cutoff $\Lambda$. In general, the new action is a very complicated non-local functional of the new fields. The Wilsonian flow equations describe how the various couplings of the effective action change as the scale $\Lambda$ is reduced. An important advantage of the Wilsonian RG over the field theory method is that the Wilsonian method can also applied to non-renormalizable field theories. In fact, condensed matter systems are usually characterized by a finite lattice spacing, which plays the role of a physical ultraviolet cutoff. Hence, there is no need to perform the continuum limit in condensed matter theory.

In the important work [5] Wegner and Houghton showed that the change of the effective action due to an *infinitesimal* change in $\Lambda$ can be described in terms of a formally exact functional differential equation. However, for practical calculations the Wegner-Houghton equation has some unpleasant features, so that it has not been widely used to solve problems of physical interest. Recently, there has been a renewed interest in exact formulations of the Wilsonian RG. Polchinski [6] noticed that a particular form of the exact Wilsonian RG equation (which is now called the Polchinski equation, see Sec. 2.1) offers a straightforward approach to proof the renormalizability of quantum field theories.

Due to its intuitive physical interpretation and its greater generality, the Wilsonian RG is the method of choice in condensed matter theory and statistical physics [3] as long as a one-loop calculation is sufficient. In some problems, however, the correct behavior of physical observables requires the knowledge of the two-loop RG $\beta$-function. An example is the asymptotic low-temperature behavior of the susceptibility of classical [7] and quantum [8] Heisenberg magnets in two dimensions. However, till now even condensed matter physicists resort to the less intuitive field theory method for two-loop calculations [9]. In fact, the opinion seems to prevail that two-loop calculations within the exact Wilsonian RG are more difficult than within the field theory method. In this work we shall show that this is not necessarily the case if one uses the irreducible version of the exact RG [10,11] in the form derived by Wetterich [12] and, independently, by Morris [13].

We consider a simple scalar field theory with a bare action

$$S\{\phi\} = S_0\{\phi\} + S_{\rm int}\{\phi\} \; , \tag{1.1}$$

where the free part is

$$S_0\{\phi\} = \frac{1}{2} \int\limits_{|\mathbf{k}|<\Lambda_0} \frac{d\mathbf{k}}{(2\pi)^D} \phi_{\mathbf{k}} G_0^{-1}(\mathbf{k}) \phi_{-\mathbf{k}} \; . \tag{1.2}$$

Here $\phi_{\mathbf{k}} = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \phi(\mathbf{r})$ are the Fourier components of a real scalar field $\phi(\mathbf{r})$ in $D$ dimensions, and

$$G_0^{-1}(\mathbf{k}) = \mathbf{k}^2 + m_0^2 \; . \tag{1.3}$$

The interaction part $S_{\rm int}\{\phi\}$ may also contain counter terms that are quadratic in the fields. Later we shall assume that it is a local function of the fields that is invariant under $\phi \rightarrow -\phi$. The one-loop RG $\beta$-function for this theory has been derived some time ago from the Polchinski equation [6] by Hughes and Liu [14]. Several strategies for calculating the RG $\beta$-function at two loops have been proposed. A derivation of the two-loop $\beta$-function entirely within the framework of the exact RG was given by Papenbrock and Wetterich [15], who used the exact evolution equation for the effective average potential and worked with a smooth cutoff in momentum space. Because sharp cutoffs in momentum space generate long-range interactions in real space, Wilson and Kogut [2] recommended to use a smooth cutoff for calculations beyond one loop. On the other hand, the one-loop $\beta$-function is most conveniently performed with the Wilsonian RG using a sharp cutoff, so that it would be useful to have a formulation of the Wilsonian RG where the same cutoff procedure can also be employed for higher order calculations. The precise relation between the orthodox field theory approach and the exact Wilsonian RG was established by Bonini *et al.* [16], who used this relation to obtain the $\beta$-function at the two-loop order. Another strategy for calculating the two-loop $\beta$-function of $\phi^4$-theory was adopted Pernici and Raciti [17], who derived the Gell-Mann Low equation from the exact RG, and then used this equation to express the $\beta$-function in terms of Wilsonian Green functions. In Ref. [13] an attempt has been made to calculate certain contributions to the two-loop coefficient of the $\beta$-function by truncating the functional differential equations of the exact RG within a momentum scale expansion; unfortunately, this truncation is uncontrolled (see also Ref. [18]) and the result is not a good approximation to the known two-loop result obtained within the field theory method [1]. Very recently Morris and Tighe [19] succeeded in calculating the two-loop $\beta$-function from the exact RG by resumming the momentum scale expansion for sharp cutoff and the derivative expansion for smooth cutoff to infinite orders.

In this work we shall show how the two-loop $\beta$-function can be obtained directly within the framework of the exact Wilsonian RG using a sharp cutoff in momentum space, without resumming the momentum scale expansion, and without any reference to the Callan-Symanzik equation. Our motivation is to demonstrate that a two-loop calculation within the exact RG is conceptually quite simple and involves only straightforward albeit tedious algebra. Our aim is to convince the reader that the exact RG is the most convenient method for performing two-loop RG calculations in condensed matter theory and statistical physics.

The rest of this paper is organized as follows: We begin in Sec.2 with a brief summary of the different versions of the exact Wilsonian RG. In Sec.3 we discuss the proper rescaling of momenta and fields in the exact RG. In Sec.4 we explicitly write down the exact flow equations for the free energy, the irreducible two-point vertex, the irreducible four-point vertex, and the irreducible six-point vertex. We also elaborate on the relation between the anomalous dimension of the field and the irreducible two-point vertex. In Sec.5 we make some general remarks concerning the structure of the RG flow. In Sec.6 we briefly re-derive the one-loop $\beta$-function. The two-loop calculation is then presented in Sec.7. Our conclusions can be found in Sec.8. Finally, in an Appendix we briefly summarize the definition of various types of generating functionals and introduce some convenient notations.

## 2 Exact flow equations

To integrate out degrees of freedom with momenta in the shell $\Lambda \leq |\mathbf{k}| < \Lambda_0$, we introduce the cutoff-dependent free propagator $G_0^{\Lambda,\Lambda_0}$, with matrix elements in momentum space given by

$$[G_0^{\Lambda,\Lambda_0}]_{\mathbf{k},\mathbf{k}'} = (2\pi)^D \delta(\mathbf{k}-\mathbf{k}') G_0^{\Lambda,\Lambda_0}(\mathbf{k}) \; , \tag{2.1}$$

where

$$G_0^{\Lambda,\Lambda_0}(\mathbf{k}) = \frac{\theta_\epsilon(|\mathbf{k}|,\Lambda) - \theta_\epsilon(|\mathbf{k}|,\Lambda_0)}{\mathbf{k}^2 + m_0^2} \; . \tag{2.2}$$

Here $\theta_\epsilon(k,\Lambda) \to 1$ for $k \gg \Lambda$, and $\theta_\epsilon(k,\Lambda) \to 0$ for $k \ll \Lambda$, such that $\theta_\epsilon(k,\Lambda)$ drops from unity to zero mainly in the interval $|k-\Lambda| \leq 2\epsilon$. We are using here the same notation as Morris [13]. For explicit calculations, we shall later on focus on the sharp cutoff case, $\lim_{\epsilon\to 0}\theta_\epsilon(k,\Lambda) = \theta(k-\Lambda)$, but at this point, we shall keep $\epsilon$ finite. There are several types of exact flow equations, corresponding to different types of correlation functions. A brief summary of

the various generating functionals and their representations in terms of functional integrals or functional differential operators is given in the Appendix. For convenience, let us now briefly summarize three types of exact flow equations. Our discussion closely follows Refs. [20,21], but we shall be more careful to keep track of all field-independent contributions to obtain the flow equation for the free energy. Apart from the three flow equations given below, two more types of flow equations should be mentioned: The Wegner-Houghton equation [5], which was derived for system with discrete momenta and sharp cutoff, is closely related to the Polchinski equation [20]. Diagrammatically these equations contain terms which are not one-particle irreducible. If one works with a sharp cutoff in momentum space, this leads to ambiguities in the thermodynamic limit, where the momenta form a continuum. Nicoll, Chang and collaborators [10] avoided these difficulties by considering the flow equation for the Legendre effective action, which generates *irreducible* correlation functions. The exact flow equation given in Sec.2.3 is equivalent to the flow equation of Ref. [11].

### *2.1 Polchinski equation for* $\mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}$

The generating functional of the amputated cutoff-connected Green functions $\mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}$ can be defined via (see Eq. (A.10))

$$e^{\mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}\{\phi\}} = e^{\frac{1}{2}(\frac{\delta}{\delta\phi}, G_0^{\Lambda,\Lambda_0}\frac{\delta}{\delta\phi})} e^{-S_{\rm int}\{\phi\}} \; . \tag{2.3}$$

Differentiating both sides with respect to $\Lambda$, we obtain the Polchinski equation [6],

$$\partial_\Lambda \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0} = \frac{1}{2}\left[ \left( \frac{\delta}{\delta\phi}, \partial_\Lambda G_0^{\Lambda,\Lambda_0} \frac{\delta}{\delta\phi}\right) \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0} + \left( \frac{\delta \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}}{\delta\phi}, \partial_\Lambda G_0^{\Lambda,\Lambda_0} \frac{\delta \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}}{\delta\phi}\right)\right] \; , \tag{2.4}$$

where $\partial_\Lambda$ on the right-hand side acts only on $G_0^{\Lambda,\Lambda_0}$. From Eq. (2.3) we see that the initial condition at $\Lambda = \Lambda_0$ is

$$\mathcal{G}_{\rm ac}^{\Lambda_0,\Lambda_0}\{\phi\} = -S_{\rm int}\{\phi\} \; . \tag{2.5}$$

Performing the functional derivatives in Eq. (2.4) in Fourier space, the Polchinski equation can also be written as

$$\partial_\Lambda \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0} = \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^D} [\partial_\Lambda G_0^{\Lambda,\Lambda_0}(\mathbf{k})] \left[ \frac{\delta^2 \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}}{\delta\phi_{\mathbf{k}} \delta\phi_{-\mathbf{k}}} + \frac{\delta \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}}{\delta\phi_{\mathbf{k}}} \frac{\delta \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}}{\delta\phi_{-\mathbf{k}}} \right] \; . \tag{2.6}$$

This equation has been used in Ref. [14] to derive the one-loop $\beta$-function for our simple scalar field theory. Let us emphasize again that diagrammatically the right-hand side of Eq. (2.6) contains terms that are one-particle reducible. This leads to technical difficulties in calculations beyond the leading order [13]. Although this problem can be avoided by expanding $\mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}\{\phi\}$ in terms of so-called Wick-ordered monomials [23], for our two-loop calculation it will be more convenient to start from the exact flow equation for the generating functional of the irreducible correlation functions, see Eq. (2.12) below.

### 2.2 Flow equation for $\mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}$

Before discussing in some detail the (for our purpose) most convenient version of the exact RG, let us mention (for completeness) the flow equation for the generating functional $\mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}\{J\}$ of the connected Green functions, which seems to be even less convenient than Eq. (2.4). From Eq. (A.7) we see that $\mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}\{J\}$ can be obtained from that of the corresponding generating functional of the amputated connected Green functions via

$$\mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}\{J\} = \mathcal{G}_{\rm ac}^{\Lambda,\Lambda_0}\{G_0^{\Lambda,\Lambda_0}J\} + \frac{1}{2}(J, G_0^{\Lambda,\Lambda_0}J) \ . \tag{2.7}$$

The exact flow equation is

$$\begin{aligned}\partial_\Lambda \mathcal{G}_{\rm c}^{\Lambda,\Lambda_0} = -\frac{1}{2}\Bigg[ &\left(\frac{\delta}{\delta J}, \partial_\Lambda (G_0^{\Lambda,\Lambda_0})^{-1}\frac{\delta}{\delta J}\right)\mathcal{G}_{\rm c}^{\Lambda,\Lambda_0} \\ &+ \left(\frac{\delta \mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}}{\delta J}, \partial_\Lambda (G_0^{\Lambda,\Lambda_0})^{-1}\frac{\delta \mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}}{\delta J}\right) + {\rm Tr}\partial_\Lambda \ln G_0^{\Lambda,\Lambda_0}\Bigg] \ ,\end{aligned} \tag{2.8}$$

with initial condition at $\Lambda = \Lambda_0$

$$\mathcal{G}_{\rm c}^{\Lambda_0,\Lambda_0}\{J\} = 0 \ . \tag{2.9}$$

This is an ill defined initial problem, so that Eq. (2.8) is not very useful in practice.

### 2.3 Flow equation for $\mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}$

The exact RG equation for the the generating functional of the irreducible vertices turns out to be the most convenient starting point for our two-loop

calculation. From Eq. (A.17) we have

$$\mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}\{\varphi\} = (\varphi, J) - \frac{1}{2}(\varphi, (G_0^{\Lambda,\Lambda_0})^{-1}\varphi) - \mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}\{J\{\varphi\}\} \,, \tag{2.10}$$

where $J$ is defined as a functional of $\varphi$ via

$$\varphi(\mathbf{r}) = \frac{\delta \mathcal{G}_{\rm c}^{\Lambda,\Lambda_0}\{J\}}{\delta J(\mathbf{r})} \ . \tag{2.11}$$

The flow equation for $\mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}\{\varphi\}$ is [12,13]

$$\partial_\Lambda \mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0} = \frac{1}{2}{\rm Tr}\left\{ \partial_\Lambda (G_0^{\Lambda,\Lambda_0})^{-1}\left[ (G_0^{\Lambda,\Lambda_0})^{-1} + \mathcal{V}^{\Lambda,\Lambda_0}\right]^{-1} - \partial_\Lambda \ln (G_0^{\Lambda,\Lambda_0})^{-1}\right\} \ , \tag{2.12}$$

where the functional $\mathcal{V}^{\Lambda,\Lambda_0}\{\varphi\}$ is the second functional derivative of $\mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}\{\varphi\}$,

$$\mathcal{V}_{\mathbf{k},\mathbf{k}'}^{\Lambda,\Lambda_0}\{\varphi\} = \frac{\delta^2 \mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}\{\varphi\}}{\delta\varphi_{\mathbf{k}}\delta\varphi_{\mathbf{k}'}} \ . \tag{2.13}$$

Unlike the authors of Refs. [12,13], we have kept track of the field-independent contributions to the right-hand side of Eq. (2.12), which determines the flow of the free energy [5]. Note that in the non-interacting limit, i.e. for $\mathcal{V}^{\Lambda,\Lambda_0} = 0$, the right-hand side of Eq. (2.12) vanishes. The initial condition at $\Lambda = \Lambda_0$ is simply

$$\mathcal{G}_{\rm ir}^{\Lambda_0,\Lambda_0}\{\varphi\} = S_{\rm int}\{\varphi\} \ . \tag{2.14}$$

It is convenient to separate at this point from Eq. (2.13) the field-independent part, which can be identified with the irreducible self-energy,

$$\mathcal{V}^{\Lambda,\Lambda_0}\{\varphi\} = \Sigma^{\Lambda,\Lambda_0} + \mathcal{U}^{\Lambda,\Lambda_0}\{\varphi\} \ , \tag{2.15}$$

where by definition $\mathcal{U}^{\Lambda,\Lambda_0}\{\varphi = 0\} = 0$. Then Eq. (2.12) can also be written as

$$\begin{aligned} \partial_\Lambda \mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0} = -\frac{1}{2}{\rm Tr}\Bigg\{ \partial_\Lambda (G_0^{\Lambda,\Lambda_0})^{-1}\Bigg[ & (G^{\Lambda,\Lambda_0})^2 \mathcal{U}^{\Lambda,\Lambda_0}\left(1 + G^{\Lambda,\Lambda_0}\mathcal{U}^{\Lambda,\Lambda_0}\right)^{-1} \\ & + (G_0^{\Lambda,\Lambda_0})^2 \Sigma^{\Lambda,\Lambda_0}\left(1 + G_0^{\Lambda,\Lambda_0}\Sigma^{\Lambda,\Lambda_0}\right)^{-1}\Bigg]\Bigg\} \ , \end{aligned} \tag{2.16}$$

where

$$G^{\Lambda,\Lambda_0} = [(G_0^{\Lambda,\Lambda_0})^{-1} + \Sigma^{\Lambda,\Lambda_0}]^{-1} \tag{2.17}$$

is the exact two-point function. Note that in momentum space

$$\mathcal{V}^{\Lambda,\Lambda_0}_{\mathbf{k},\mathbf{k}'}\{\varphi\} = (2\pi)^D \delta(\mathbf{k}+\mathbf{k}')\Sigma^{\Lambda,\Lambda_0}(\mathbf{k}) + \mathcal{U}^{\Lambda,\Lambda_0}_{\mathbf{k},\mathbf{k}'}\{\varphi\} \ , \tag{2.18}$$

$$G^{\Lambda,\Lambda_0}_{\mathbf{k},\mathbf{k}'} = (2\pi)^D \delta(\mathbf{k}+\mathbf{k}') \frac{\theta_\epsilon(|\mathbf{k}|,\Lambda) - \theta_\epsilon(|\mathbf{k}|,\Lambda_0)}{\mathbf{k}^2 + m_0^2 + \Sigma^{\Lambda,\Lambda_0}(\mathbf{k})} \ . \tag{2.19}$$

For our two-loop calculation, it will be useful to take the sharp cutoff limit of Eq. (2.16). In this limit $\theta_\epsilon(k,\Lambda) \to \theta(k-\Lambda)$. Performing the trace in momentum space, Eq. (2.16) reduces to [13]

$$\begin{aligned} \partial_\Lambda \mathcal{G}^{\Lambda,\Lambda_0}_{\rm ir} = & -\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^D} \frac{\delta(|\mathbf{k}|-\Lambda)}{\Lambda^2 + m_0^2 + \Sigma^{\Lambda,\Lambda_0}(\mathbf{k})} \left[ \mathcal{U}^{\Lambda,\Lambda_0} \left( 1 + G^{\Lambda,\Lambda_0}\mathcal{U}^{\Lambda,\Lambda_0} \right)^{-1} \right]_{\mathbf{k},-\mathbf{k}} \\ & -\frac{V}{2}\int \frac{d\mathbf{k}}{(2\pi)^D} \delta(|\mathbf{k}|-\Lambda) \ln\left[ \frac{\Lambda^2 + m_0^2 + \Sigma^{\Lambda,\Lambda_0}(\mathbf{k})}{\Lambda^2 + m_0^2} \right] \ , \end{aligned} \tag{2.20}$$

where $V$ is the volume of the system. In deriving Eq. (2.20) we have ignored possible ambiguities that might arise from functions $\theta(0)$ contained in the term $\mathcal{U}^{\Lambda,\Lambda_0}\left(1 + G^{\Lambda,\Lambda_0}\mathcal{U}^{\Lambda,\Lambda_0}\right)^{-1}$. See Ref. [13] and the remark in Sec.4.3 for a further discussion of this point.

## 3 Rescaling momenta and fields

As it stands, Eq. (2.20) describes the change in the Legendre effective action due to the elimination of degrees of freedom with momenta in the interval $[\Lambda, \Lambda_0]$. A complete Wilsonian RG transformation includes also a rescaling of momenta and fields. A convenient way of including these rescaling transformations into the above flow equations it to rewrite them in terms of dimensionless variables by multiplying all quantities by appropriate powers of $\Lambda$. By taking the $\Lambda$-derivative while keeping the dimensionless variables constant, the rescalings are implicitly carried out [20]. In our case, it is useful to introduce dimensionless variables as follows,

$$\mathbf{k} = \Lambda \mathbf{q} \ \ , \ \ \Lambda = \Lambda_0 e^{-t} \ \ , \ \ \varphi_{\mathbf{k}} = \Lambda^{D^\varphi - D}\sqrt{Z_t}\tilde{\varphi}_{\mathbf{q}} \ . \tag{3.1}$$

Here $D^\varphi = \frac{1}{2}(D-2)$ is the scaling dimension of the field $\varphi(\mathbf{r})$ in real space (so that $D^\varphi - D$ is the scaling dimension of its Fourier transform $\varphi_{\mathbf{k}}$). The

scale-dependent dimensionless factor $Z_t$ is the wave-function renormalization factor, which is related to the anomalous dimension $\eta_t$ of the field via

$$\eta_t = -\partial_t \ln Z_t \ . \tag{3.2}$$

At this point the introduction of $Z_t$, which cannot be derived from dimensional analysis, seems to be a rather ad hoc procedure. However, in general it is necessary to introduce such a factor, because otherwise it may happen that the RG transformation never reaches a fixed point [4,5,20]. Given Eq. (3.1), it is (almost) natural to define

$$\mathcal{G}_{\mathrm{ir}}^{\Lambda,\Lambda_0}\{\varphi_{\mathbf{k}}\} - \frac{N_t}{2} \ln Z_t = \mathcal{G}_t\{\tilde{\varphi}_{\mathbf{q}}\} \ , \tag{3.3}$$

$$\mathcal{U}^{\Lambda,\Lambda_0}\{\varphi_{\mathbf{k}}\} = \Lambda^{-2D^{\varphi}} Z_t^{-1} \mathcal{U}_t\{\tilde{\varphi}_{\mathbf{q}}\} \ , \tag{3.4}$$

$$G^{\Lambda,\Lambda_0} = \Lambda^{2D^{\varphi}} Z_t G_t \ , \tag{3.5}$$

$$\mathbf{k}^2 + m_0^2 + \Sigma^{\Lambda,\Lambda_0}(\mathbf{k}) = \Lambda^2 Z_t r_t(|\mathbf{q}|) \ , \tag{3.6}$$

where have used the fact that $\Sigma^{\Lambda,\Lambda_0}(\mathbf{k})$ depends only on $|\mathbf{k}|$. Here $N_t$ is the number of Fourier components in a system with ultraviolet cutoff $\Lambda = \Lambda_0 e^{-t}$,

$$N_t = V \int \frac{d\mathbf{k}}{(2\pi)^D} \theta(\Lambda_0 e^{-t} - |\mathbf{k}|) = \frac{K_D}{D} V \Lambda_0^D e^{-Dt} \equiv N_0 e^{-Dt} \ , \tag{3.7}$$

where $K_D$ is the volume of the unit sphere in $D$ dimensions divided by $(2\pi)^D$,

$$K_D = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \ . \tag{3.8}$$

The constant $\frac{N_t}{2} \ln Z_t$ in Eq. (3.3) modifies the scaling of the free energy. This term is due to the fact that the anomalous field rescaling leads also to a change in the integration measure, as explained in detail by Wegner and Houghton [5]. In our case, field rescaling $\phi_{\mathbf{k}} \to \Lambda^{D^{\varphi}-D} Z_t \phi_{\mathbf{k}}$ modifies the generating functional of the cutoff connected Green functions as

$$e^{\mathcal{G}_c^{\Lambda,\Lambda_0}} \to Z_t^{\frac{N_t}{2}} e^{\mathcal{G}_c^{\Lambda,\Lambda_0}} \ , \tag{3.9}$$

so that

$$\mathcal{G}_c^{\Lambda,\Lambda_0} \to \mathcal{G}_c^{\Lambda,\Lambda_0} + \frac{N_t}{2} \ln Z_t \ , \tag{3.10}$$

and hence, from Eq. (A.17)

$$\mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0} \to \mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0} - \frac{N_t}{2} \ln Z_t \ . \tag{3.11}$$

Substituting the above rescaled variables into Eq. (2.20) the left-hand-side becomes (after multiplication with a factor of $\Lambda$)

$$\Lambda \partial_\Lambda \mathcal{G}_{\rm ir}^{\Lambda,\Lambda_0}\{\varphi_{\mathbf{k}}\} = -\partial_t \mathcal{G}_t\{\tilde{\varphi}_{\mathbf{q}}\} + \hat{\mathcal{R}}_s \mathcal{G}_t\{\tilde{\varphi}_{\mathbf{q}}\} + \frac{N_t}{2}\left[\eta_t + D \ln Z_t\right] \ , \tag{3.12}$$

where $\hat{\mathcal{R}}_s$ is the functional differential operator describing the rescaling of fields and momenta [5,20]

$$\begin{aligned}\hat{\mathcal{R}}_s &= -\int \frac{d\mathbf{q}}{(2\pi)^D} \tilde{\varphi}_{\mathbf{q}} \mathbf{q} \cdot \nabla_{\mathbf{q}} \frac{\delta}{\delta \tilde{\varphi}_{\mathbf{q}}} - D_t^{\varphi} \int \frac{d\mathbf{q}}{(2\pi)^D} \tilde{\varphi}_{\mathbf{q}} \frac{\delta}{\delta \tilde{\varphi}_{\mathbf{q}}} \\ &= \int \frac{d\mathbf{q}}{(2\pi)^D} \left[(\mathbf{q} \cdot \nabla_{\mathbf{q}} \tilde{\varphi}_{\mathbf{q}}) + (D - D_t^{\varphi}) \tilde{\varphi}_{\mathbf{q}}\right] \frac{\delta}{\delta \tilde{\varphi}_{\mathbf{q}}} \ . \end{aligned} \tag{3.13}$$

Here

$$D_t^{\varphi} = D^{\varphi} + \frac{\eta_t}{2} = \frac{1}{2}(D - 2 + \eta_t) \tag{3.14}$$

is the full dimension of the renormalized field $\tilde{\varphi}$ (in real space). Performing the angular integration in Eq. (2.20) in terms of $D$-dimensional spherical coordinates, we finally obtain the exact flow equation

$$\partial_t \mathcal{G}_t = \hat{\mathcal{R}}_s \mathcal{G}_t + \frac{K_D}{2 r_t(1)} \left\langle \left[\mathcal{U}_t (1 + G_t \mathcal{U}_t)^{-1}\right]_{\hat{\mathbf{q}},-\hat{\mathbf{q}}} \right\rangle_{\hat{\mathbf{q}}} + \frac{N_t}{2}\left[\eta_t + D \ln\left(\frac{r_t(1)}{r_0(1)}\right)\right] \ . \tag{3.15}$$

Here $\langle \ldots \rangle_{\hat{\mathbf{q}}}$ means averaging over all directions of the $D$-dimensional unit vector $\hat{\mathbf{q}}$. Note that $G_t$ is a diagonal matrix in momentum space with matrix elements

$$[G_t]_{\mathbf{q},\mathbf{q}'} = (2\pi)^D \delta(\mathbf{q} + \mathbf{q}') G_t(\mathbf{q}) \ , \tag{3.16}$$

$$G_t(\mathbf{q}) = \frac{\theta(|\mathbf{q}| - 1) - \theta(|\mathbf{q}| - e^t)}{r_t(|\mathbf{q}|)} \ , \tag{3.17}$$

where $r_t(q)$ is defined in Eq. (3.6). The initial values at $t = 0$ are by construction

$$G_{t=0}(\mathbf{q}) = 0 \ \ , \ \ Z_{t=0} = 1 \ \ , \ \ r_{t=0}(q) = q^2 + \left( \frac{m_0}{\Lambda_0} \right)^2 \ . \tag{3.18}$$

Eq. (3.15) is the central result of this section. At a fixed point of the RG $\mathcal{G}_t$ become stationary,

$$\partial_t \mathcal{G}_t \{ \tilde{\varphi}_{\mathbf{q}} \} = 0 \ . \tag{3.19}$$

Of course, usually it is impossible to solve Eq. (3.15) exactly, so that we have to resort to approximations.

## 4 Exact flow equations for the irreducible vertices

To generate an expansion in the number of loops, let us we expand the functional $\mathcal{G}_t\{\tilde{\varphi}_{\mathbf{q}}\}$ in powers of Fourier components $\tilde{\varphi}_{\mathbf{q}}$ of the field,

$$\begin{aligned} \mathcal{G}_t\{\tilde{\varphi}_{\mathbf{q}}\} = \Gamma_t^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d\mathbf{q}_1}{(2\pi)^D} \cdots \int \frac{d\mathbf{q}_n}{(2\pi)^D} (2\pi)^D \delta(\mathbf{q}_1 + \ldots + \mathbf{q}_n) \\ \times \Gamma_t^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) \tilde{\varphi}_{\mathbf{q}_1} \ldots \tilde{\varphi}_{\mathbf{q}_n} \ . \end{aligned} \tag{4.1}$$

The functional $\mathcal{U}_t$ on the right-hand side of Eq. (3.15) can be written as

$$\begin{aligned} [\mathcal{U}_t\{\tilde{\varphi}_{\mathbf{q}}\}]_{\mathbf{q},\mathbf{q}'} = \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d\mathbf{q}_1}{(2\pi)^D} \cdots \int \frac{d\mathbf{q}_n}{(2\pi)^D} (2\pi)^D \delta(\mathbf{q}_1 + \ldots + \mathbf{q}_n) \\ \times \Gamma_t^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) \frac{\delta^2}{\delta \tilde{\varphi}_{\mathbf{q}} \delta \tilde{\varphi}_{\mathbf{q}'}} \left[ \tilde{\varphi}_{\mathbf{q}_1} \ldots \tilde{\varphi}_{\mathbf{q}_n} \right] \ . \end{aligned} \tag{4.2}$$

We now substitute Eqs.(4.1) and (4.2) into Eq. (3.15) and expand

$$\mathcal{U}_t (1 + G_t \mathcal{U}_t)^{-1} = \mathcal{U}_t - \mathcal{U}_t G_t \mathcal{U}_t + \mathcal{U}_t G_t \mathcal{U}_t G_t \mathcal{U}_t - \ldots \ . \tag{4.3}$$

Collecting all terms with the same powers of the fields on both sides of Eq. (3.15), we obtain an infinite hierarchy of flow equations for the irreducible $n$-point vertices $\Gamma_t^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n)$. If the bare action is invariant under $\varphi \to -\varphi$, only the vertices with even $n$ are finite. It turns out that for the two-loop calculation of the RG $\beta$-function it is sufficient to truncate the hierarchy by setting $\Gamma_t^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) = 0$ for $n \geq 8$. We now explicitly give the exact

flow equations for the irreducible vertices that are required for the two-loop calculation.

### *4.1 Free energy*

The flow equation for $\Gamma_t^{(0)}$ describes the flow of the interaction correction to the free energy. From the field-independent part of our exact flow equation (3.15) we find

$$\partial_t \Gamma_t^{(0)} = \frac{N_t}{2} \left[ \eta_t + D \ln \left( \frac{r_t(1)}{r_0(1)} \right) \right] . \tag{4.4}$$

The initial condition is

$$\Gamma_{t=0}^{(0)} = 0 . \tag{4.5}$$

Setting $\Gamma_t^{(0)} = N_t f_t$ and using $N_t = N_0 e^{-Dt}$ (see Eq. (3.7)), we obtain the flow equation for the interaction correction to the free energy per Fourier component,

$$\partial_t f_t = D f_t + \frac{\eta_t}{2} + \frac{D}{2} \ln \left( \frac{r_t(1)}{r_0(1)} \right) . \tag{4.6}$$

Note that in the corresponding flow equation for the potential $v_0$ given by Wegner and Houghton (see Eq. (3.9) of Ref. [5]) the term $\eta_t$ is replaced by $\eta - 2$, where $\eta$ is assumed to be independent of $t$. The extra $-2$ is due to the fact that the potential $v_0$ introduced in Ref. [5] represents the total free energy, while our $\Gamma_t^{(0)}$ contains only the interaction correction to the free energy. Hence for a non-interacting system our $\Gamma_t^{(0)}$ does not flow and remains identically zero. The solution of Eq. (4.6) with the correct initial condition is

$$f_t = \frac{1}{2} \int_0^t dt' e^{D(t-t')} \left[ \eta_{t'} + D \ln \left( \frac{r_{t'}(1)}{r_0(1)} \right) \right] . \tag{4.7}$$

### *4.2 Irreducible two-point vertex and self-energy*

From Eq. (3.15) we obtain the following exact flow equation for the dimensionless irreducible two-point vertex,

$$\partial_t \Gamma_t^{(2)}(\mathbf{q},-\mathbf{q}) = (2-\eta_t - \mathbf{q}\cdot\nabla_{\mathbf{q}})\,\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q}) + \frac{K_D}{2r_t(1)}\left\langle \Gamma_t^{(4)}(\mathbf{q},-\mathbf{q},\hat{\mathbf{q}}',-\hat{\mathbf{q}}')\right\rangle_{\hat{\mathbf{q}}'} \ . \tag{4.8}$$

The initial condition at $t=0$ is determined by the interaction part of the bare action $S_{\rm int}\{\varphi\}$, see Eq. (2.14). If $S_{\rm int}\{\varphi\}$ does not have any term quadratic in the fields, then the initial condition is $\Gamma_{t=0}^{(2)}(\mathbf{q},-\mathbf{q})=0$. However, to reach the Gaussian fixed point in $D\geq 4$ it is necessary to fine tune the bare action such that the system flows along the critical surface. As explained in detail in Sec.5, in this case the initial value $\Gamma_{t=0}^{(2)}(0,0)$ becomes a function of the bare four-point vertex (see Fig.1 and Eq. (5.5)). Of course, such quadratic term in the fields can also be absorbed into a redefinition of the Gaussian part of the action, but we find it more convenient to include all interaction-dependent terms in $S_{\rm int}\{\varphi\}$.

By definition, our dimensionless parameter $r_t(q)$ is related to the irreducible two-point vertex $\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q})$ via

$$r_t(q) = Z_t r_0(q) + \Gamma_t^{(2)}(\mathbf{q},-\mathbf{q}) \ , \tag{4.9}$$

implying

$$[\partial_t - 2 + \eta_t + \mathbf{q}\cdot\nabla_{\mathbf{q}}][\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q}) - r_t(q)] = 2Z_t\frac{m_0^2}{\Lambda_0^2} \ . \tag{4.10}$$

Hence, up to a re-definition of the four-point vertex by a momentum-independent constant proportional to $m_0^2/\Lambda_0^2$, the flow equations for $r_t(q)$ and $\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q})$ are identical. Note that the authors of Ref. [11] prefer to work with $r_t(q)$ (their flow equation for $r_t(q)$ is thus equivalent to our Eq. (4.8), except that these authors replace $\eta_t$ by its fixed point value). In this work we find it more convenient to parameterize the two-point function in terms of its irreducible part $\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q})$.

Physically it is clear that the irreducible two-point function should uniquely determine the anomalous dimension $\eta_t = -\partial_t \ln Z_t$ of the field. Hence, $\eta_t$ on the right-hand side of Eq. (4.8) implicitly depends on $\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q})$. The precise relation becomes more transparent when we rewrite Eq. (4.8) in terms of the dimensionless irreducible self-energy $\sigma_t(q)$ defined by

$$\sigma_t(q) \equiv \Lambda^{-2}\Sigma^{\Lambda,\Lambda_0}(\Lambda\mathbf{q}) = Z_t^{-1}\Gamma_t^{(2)}(\mathbf{q},-\mathbf{q}) \ , \tag{4.11}$$

where $\Lambda = \Lambda_0 e^{-t}$. Using $\partial_t Z_t = -Z_t\eta_t$, it is easy to show that Eq. (4.8) is equivalent with

$$\partial_t \sigma_t(q) = (2 - \mathbf{q} \cdot \nabla_{\mathbf{q}})\, \sigma_t(q) + \frac{K_D}{2 Z_t^2 [r_0(1) + \sigma_t(1)]} \left\langle \Gamma_t^{(4)}(\mathbf{q}, -\mathbf{q}, \hat{\mathbf{q}}', -\hat{\mathbf{q}}') \right\rangle_{\hat{\mathbf{q}}'} \ . \tag{4.12}$$

The wave-function renormalization $Z_t$ on the right-hand side is now fixed by demanding that [22]

$$q^2 + \sigma_t(q) - \sigma_t(0) = Z_t^{-1} q^2 + O(q^3) \ , \tag{4.13}$$

which implies

$$Z_t^{-1} = 1 + \left. \frac{\partial \sigma_t(q)}{\partial (q^2)} \right|_{q^2=0} \ . \tag{4.14}$$

From the definition Eq. (4.11) it is then easy to show that $Z_t$ is related to the irreducible two-point vertex via

$$Z_t = 1 - \left. \frac{\partial \Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q})}{\partial (q^2)} \right|_{q^2=0} \ . \tag{4.15}$$

Using Eq. (4.14) to eliminate $Z_t$ from Eq. (4.12), we obtain the following non-linear partial differential equation for $\sigma_t(q)$,

$$\partial_t \sigma_t(q) = (2 - \mathbf{q} \cdot \nabla_{\mathbf{q}})\, \sigma_t(q) + \frac{K_D}{2[r_0(1) + \sigma_t(1)]} \left[ 1 + \left. \frac{\partial \sigma_t(q)}{\partial (q^2)} \right|_{q^2=0} \right]^2 \left\langle \Gamma_t^{(4)}(\mathbf{q}, -\mathbf{q}, \hat{\mathbf{q}}', -\hat{\mathbf{q}}') \right\rangle_{\hat{\mathbf{q}}'} \ . \tag{4.16}$$

Given a solution of Eq. (4.16) with appropriate initial condition, the anomalous dimension of the field can be obtained from Eq. (4.14). Note, however, that the condition (3.19) that the RG has a fixed point requires

$$\partial_t \Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q}) = 0 \ , \tag{4.17}$$

which according to Eq. (4.11) means that

$$\partial_t \sigma_t(q) = \eta \sigma_t(q) \ , \tag{4.18}$$

where $\eta$ is the value of $\eta_t$ at the RG fixed point. Thus, for a system with a finite anomalous dimension the irreducible self-energy is not stationary at a fixed point of the RG. This is the reason why Eq. (4.8) is more useful than Eq. (4.12) if we are interested in possible fixed points of the RG. Keeping

in mind that $\partial_t \sigma_t(q)$ on the left-hand-side of Eq. (4.18) can be replaced by the right-hand side of Eq. (4.16), we see that Eq. (4.18) can also be viewed as a non-linear eigenvalue equation [20,24]. From this point of view it is not surprising that a RG fixed point can only be reached for certain discrete values of $\eta$ [4].

### *4.3 Irreducible four-point vertex*

The irreducible four-point vertex satisfies the following flow equation,

$$
\begin{aligned}
\partial_t \Gamma_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) &= \left[ 4 - D - 2\eta_t - \sum_{i=1}^{4} \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} \right] \Gamma_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \\
&+ \frac{K_D}{2r_t(1)} \left\langle \Gamma_t^{(6)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \right\rangle_{\hat{\mathbf{q}}} \\
&- \frac{K_D}{r_t(1)} \Big\langle \Gamma_t^{(4)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}} - \mathbf{q}_1 - \mathbf{q}_2, \mathbf{q}_1, \mathbf{q}_2) G_t(\hat{\mathbf{q}} + \mathbf{q}_1 + \mathbf{q}_2) \\
&\quad \times \Gamma_t^{(4)}(\hat{\mathbf{q}} + \mathbf{q}_1 + \mathbf{q}_2, -\hat{\mathbf{q}}, \mathbf{q}_3, \mathbf{q}_4) + (\mathbf{q}_2 \leftrightarrow \mathbf{q}_3) + (\mathbf{q}_2 \leftrightarrow \mathbf{q}_4) \Big\rangle_{\hat{\mathbf{q}}} .
\end{aligned}
\tag{4.19}
$$

As pointed out by Morris [13], this equation is only valid if none of the combinations of external momenta $\mathbf{q}_1 + \mathbf{q}_2$, $\mathbf{q}_1 + \mathbf{q}_3$, and $\mathbf{q}_1 + \mathbf{q}_4$ that enter the propagators $G_t(\hat{\mathbf{q}} + \mathbf{q}_1 + \mathbf{q}_i)$ vanishes. Otherwise these expressions contain the function $\theta(0)$, which is ambiguous. However, in physical problems where nothing drastic happens at vanishing momenta these special points in momentum space should not matter in the calculation of any observable. In this case we may simply ignore this problem, which we shall do from now on. On the other hand, if the field acquires a vacuum expectation value so that its $\mathbf{q} = 0$ Fourier component is finite, the sharp cutoff limit has to be taken more carefully, or the vacuum expectation value has to be treated separately [13].

### *4.4 Irreducible six-point vertex*

Eq. (3.15) implies the following flow equation for the irreducible six-point vertex,

$$
\begin{aligned}
\partial_t \Gamma_t^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) &= \left[ 6 - 2D - 3\eta_t - \sum_{i=1}^{6} \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} \right] \Gamma_t^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) \\
&+ \frac{K_D}{2r_t(1)} \left\langle \Gamma_t^{(8)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, \mathbf{q}_1, \ldots, \mathbf{q}_6) \right\rangle_{\hat{\mathbf{q}}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{K_D}{r_t(1)} \sum_{\{I_1,I_2\}}^{15 \text{ terms}} \left\langle \Gamma_t^{(4)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}} - \mathbf{Q}_1, I_1) G_t(\hat{\mathbf{q}} + \mathbf{Q}_1) \Gamma_t^{(6)}(\hat{\mathbf{q}} + \mathbf{Q}_1, -\hat{\mathbf{q}}, I_2) \right\rangle_{\hat{\mathbf{q}}} \\
&+\frac{K_D}{r_t(1)} \sum_{\{I_1,I_2\},I_3}^{45 \text{ terms}} \Big\langle \Gamma_t^{(4)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}} - \mathbf{Q}_1, I_1) G_t(\hat{\mathbf{q}} + \mathbf{Q}_1) \Gamma_t^{(4)}(\hat{\mathbf{q}} + \mathbf{Q}_1, -\hat{\mathbf{q}} + \mathbf{Q}_2, I_3) \\
&\qquad \times G_t(\hat{\mathbf{q}} - \mathbf{Q}_2) \Gamma_t^{(4)}(\hat{\mathbf{q}} - \mathbf{Q}_2, -\hat{\mathbf{q}}, I_2) \Big\rangle_{\hat{\mathbf{q}}} . \qquad (4.20)
\end{aligned}
$$

We are using here the same notation as Morris [13]: $\mathbf{Q}_i = \sum_{\mathbf{q}_k \in I_i} \mathbf{q}_k$, and $\sum_{\{I_1,I_2\},I_3,\ldots I_m}$ means a sum over all disjoint subsets, $I_i \cap I_j = \emptyset$ such that $\cup_{i=1}^m I_i = \{\mathbf{q}_1, \ldots \mathbf{q}_n\}$. The symbol $\{I_1, I_2\}$ means that this pair is counted only once. The 15 terms of the sum involving the combination $\Gamma^{(4)}\Gamma^{(6)}$ correspond to the $15 = \binom{6}{4}$ possibilities of choosing distinct subsets of four momenta out of the six available momenta. The 45 terms in the last sum of Eq. (4.20) correspond to one half of the $90 = \binom{6}{4}\binom{4}{2}$ possibilities of picking subsets of four momenta out of a set of six momenta, and then picking again subsets of two out of these four momenta.

## 5 Relevant and irrelevant couplings

Before embarking on the two-loop calculation of the RG $\beta$-function, we should identify the relevant and irrelevant couplings.

### *5.1 Relevant couplings*

In $D \geq 4$ the only fixed point of the RG flow is the Gaussian fixed point, where all irreducible $n$-point vertices with $n > 2$ vanish. Following the usual jargon, we call all couplings with a positive or vanishing scaling dimension *relevant couplings*. For $D > 4$ there are only two relevant couplings. The first is the momentum-independent part of the irreducible two-point vertex,

$$
\mu_t \equiv \Gamma_t^{(2)}(0,0) = Z_t \sigma_t(0) . \qquad (5.1)
$$

This coupling is strongly relevant and has scaling dimension +2. The second relevant coupling is the coefficient of the term of order $q^2$ in the expansion of $\Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q})$ for small dimensionless wave-vectors $\mathbf{q}$,

$$
c_t \equiv \left. \frac{\partial \Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q})}{\partial (q^2)} \right|_{q^2=0} = 1 - Z_t , \qquad (5.2)
$$

see Eq. (4.15). The scaling dimension of $c_t$ vanishes. Let us emphasize that in general $c_t$ does not vanish at the Gaussian fixed point, but approaches a finite value which depends on the bare interaction, see Fig.2 below. Instead of $\mu_t$ and $c_t$, we could also parameterize the two relevant couplings in terms of $\sigma_t(0)$ and $Z_t$, but we prefer to work with $\mu_t$ and $c_t$, because these quantities are directly related to the derivatives of our dimensionless two-point vertex $\Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q})$. In $D \leq 4$ the momentum-independent part of the four-point vertex

$$g_t \equiv \Gamma_t^{(4)}(0,0,0,0) \tag{5.3}$$

is also relevant. The scaling dimension of $g_t$ is positive in $D < 4$ and vanishes in $D = 4$. All other couplings are irrelevant in $D \geq 4$, so that the RG trajectory in the infinite-dimensional parameter space of all couplings rapidly flows approaches the three-dimensional subspace spanned by the relevant couplings $\{\mu_t, c_t, g_t\}$. As explained in detail by Polchinski [6], for sufficiently large $t$ the irrelevant couplings can then be expanded in powers of the relevant couplings.

From now on we shall consider the flow on the critical surface in $D \geq 4$, which is the infinite-dimensional manifold in coupling space which flows into the Gaussian fixed point under the RG transformation. Recall that in $D < 4$ the Gaussian fixed point becomes unstable, and a non-trivial fixed point emerges, the Wilson-Fisher fixed point [2,3]. For simplicity, we shall assume a vanishing bare mass ($m_0 = 0$), so that $r_0(q) = q^2$. The projection of the RG flow onto the plane spanned by $g_t$ and $\mu_t$ is shown in Fig.1. Obviously, at the Gaussian fixed point the mass renormalization $\mu_t$ vanishes, so that this fixed point represents a system at criticality. For sufficiently small $g_t$ the critical trajectory can be approximated by a straight line through the origin in the $g$-$\mu$-plane,

$$\mu_t = \gamma^{(1)} g_t + O(g_t^2) \; . \tag{5.4}$$

Note that Eq. (5.4) implies that, in order to reach the Gaussian fixed point, the initial value of the two-point vertex should be adjusted such that

$$\mu_{t=0} \equiv \Gamma_{t=0}^{(2)}(0,0) = \gamma^{(1)} \Gamma_{t=0}^{(4)}(0,0,0,0) \; . \tag{5.5}$$

To determine the numerical constant $\gamma^{(1)}$, we use the fact that according to Eq. (4.8)

$$\partial_t \mu_t = 2\mu_t + \frac{K_D}{2 r_0(1)} g_t + O(g_t^2) \; , \tag{5.6}$$

while Eq. (4.19) implies to leading order

$$\partial_t g_t = (4 - D) g_t + O(g_t^2) \; . \tag{5.7}$$

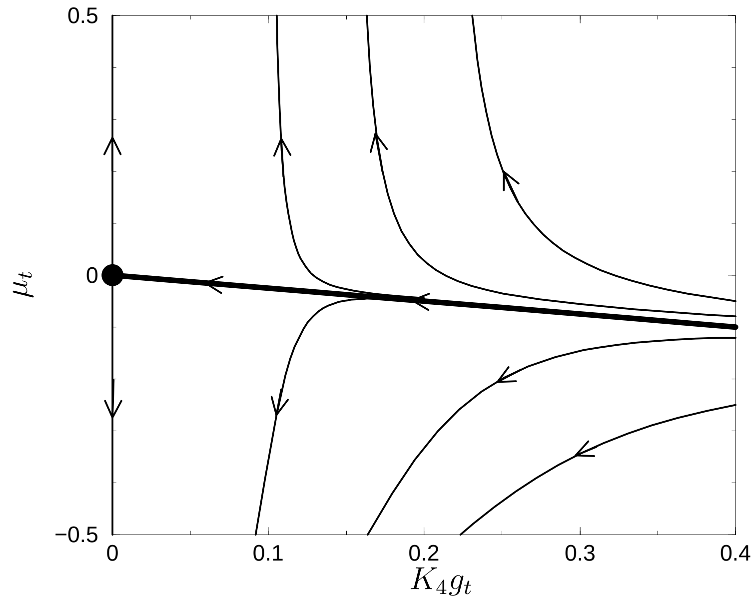


Fig. 1. One-loop RG-flow for $D = 4$ in the $g$-$\mu$-plane. Defining $\tilde{g}_t = K_D g_t$, the one-loop flow is determined by $\partial_t \mu_t = 2\mu_t + \frac{1}{2}\tilde{g}_t/(1 + \mu_t)$ and $\partial_t \tilde{g}_t = (4 - D)\tilde{g}_t - \frac{3}{2}\tilde{g}_t^2/(1 + \mu_t)^2$. The thick black line is the linear approximation for the critical trajectory ($\mu_t = -\frac{1}{4}\tilde{g}_t$, see Eq. (5.4)), which flows towards the Gaussian fixed point (black dot). To reach this fixed point, the initial values $g_{t=0}$ and $\mu_{t=0}$ must be fine tuned to lie on the critical trajectory.

Here we have anticipated that the lowest corrections to $\eta_t$ and $c_t = 1 - Z_t$ involve at least two powers of $g_t$. Keeping in mind that $r_0(1) = 1$, substituting Eq. (5.4) into Eq. (5.6) and using Eq. (5.7), we find

$$\gamma^{(1)} = -\frac{K_D}{2(D-2)} \ . \tag{5.8}$$

For the two-loop calculation of the $\beta$-function presented below the linear accuracy given in Eq. (5.4) turns out to be sufficient.

Having adjusted the strongly relevant coupling $\mu_t$ such that we reach the Gaussian fixed point, we are left with two relevant couplings, $g_t$ and $c_t$. Of course, in addition we have infinitely many irrelevant couplings, which are given by the higher order momentum-dependence of the two-point and four-point vertices, and the higher irreducible vertices. However, the irrelevant couplings can be expanded in powers of the relevant couplings, which is the key to perform a two-loop calculation within the flow equation formalism. Hence, in general we expect that the projection of the exact RG flow onto the subspace spanned by couplings $g_t$ and $c_t$ can be described by equations of the

form

$$\partial_t c_t = A(g_t, c_t) \quad , \quad \partial_t g_t = B(g_t, c_t) \; , \tag{5.9}$$

with some dimensionless functions $A(g, c)$ and $B(g, c)$.

### *5.2 Irrelevant couplings*

Because the RG trajectory rapidly approaches the manifold spanned by the relevant couplings, all couplings with negative scaling dimension (the irrelevant couplings) become local functions of the relevant couplings, which are independent of the initial conditions at $t = 0$. This is explained in detail in the seminal paper by Polchinski [6]. From the exact flow equation (3.15) for the irreducible vertices it is easy to see that for small $g_t$ and large $t$ the leading behavior of the irrelevant coupling functions is

$$\Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q}) - \mu_t - c_t q^2 \sim g_t^2 \gamma^{(2)}(q) + O(g_t^3) \; . \tag{5.10}$$

$$\Gamma_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) - g_t \sim g_t^2 \gamma^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + O(g_t^3) \; . \tag{5.11}$$

$$\Gamma_t^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) \sim g_t^{n/2} \gamma^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) + O(g_t^{(n+2)/2}) \; . \tag{5.12}$$

Here the $\gamma^{(n)}(\mathbf{q}_1, \ldots, \mathbf{q}_n)$ are $t$-independent functions. Note that the leading corrections are independent of $c_t$. This is due to the fact that the leading term on the right-hand side of the flow equation (5.9) for $c_t$ turns out to be proportional to $g_t^2$, see Eq. (7.25) below. For a two-loop calculation within the exact RG we need to know the functions $\gamma^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ and $\gamma^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6)$.

## 6 One-loop $\beta$-function

The one-loop $\beta$-function has been calculated from Polchinski's flow equation by Hughes and Liu [14]. To this order, we may truncate the system of exact flow equation by setting $\Gamma_t^{(n)} = 0$ for $n \geq 6$. Furthermore, for a one-loop calculation the momentum-dependence of all couplings can be ignored. In particular, the wave-function renormalization can be neglected at one-loop order, so that $Z_t = 1$ and hence $c_t = \eta_t = 0$. Obviously, the function $A(g_t, c_t)$ in Eq. (5.9) vanishes to this order. From the exact flow equation (4.19) we obtain for the momentum-independent part of the four-point vertex,

$$\partial_t g_t = (4 - D) g_t - 3 K_D \langle G_t(\hat{\mathbf{q}}) \rangle_{\hat{\mathbf{q}}} \, g_t^2 + O(g_t^3, c_t g_t^2) \; , \tag{6.1}$$

where $G_t(\mathbf{q})$ is defined in Eq. (3.17). Note that by definition $G_{t=0}(\mathbf{q}) = 0$, but for $t > 0$ and $|\mathbf{q}| \leq 1$ the ultraviolet cutoff $\theta(|\mathbf{q}| - e^t)$ in Eq. (3.17) can be neglected, so that to leading order we may approximate

$$G_t(\mathbf{q}) \approx \frac{\theta(|\mathbf{q}| - 1)}{r_0(|\mathbf{q}|)} \equiv C(\mathbf{q}) \quad , \quad t > 0 \quad , \quad |\mathbf{q}| \leq 1 \; . \tag{6.2}$$

Using

$$\langle C(\hat{\mathbf{q}}) \rangle_{\hat{\mathbf{q}}} = \frac{1}{2} \; , \tag{6.3}$$

we finally obtain for the right-hand sides $A(g_t, c_t)$ and $B(g_t, c_t)$ of the flow equations (5.9) for the relevant couplings

$$A(g_t, c_t) = O(g_t^2) \; , \tag{6.4}$$

$$B(g_t, c_t) = (4 - D) g_t - \frac{3}{2} K_D g_t^2 + O(g_t^3) \; . \tag{6.5}$$

Eq. (6.5) is the known one-loop $\beta$-function of massless $\phi^4$-theory [1]. Recall that in four dimensions $K_4 = 2/(4\pi)^2$.

## 7 Two-loop $\beta$-function

To obtain the flow equation for $g_t$ at the two-loop order, we substitute Eqs.(5.11) and (5.12) into the exact flow equation (4.19) for the irreducible four-point vertex and collect all terms up to order $g_t^3$. After a straightforward calculation we obtain for the generalized $\beta$-function defined in Eq. (5.9),

$$\begin{aligned} B(g_t, c_t) = & (4 - D - 2\eta_t) g_t - \frac{3}{2} K_D g_t^2 \\ & + 3 K_D g_t^3 \left[ \gamma^{(1)} - \left\langle \gamma^{(4)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, 0, 0) \right\rangle_{\hat{\mathbf{q}}} + \frac{1}{6} \left\langle \gamma^{(6)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, 0, 0, 0, 0) \right\rangle_{\hat{\mathbf{q}}} \right] \\ & + O(g_t^4) \; . \end{aligned} \tag{7.1}$$

Recall that by definition

$$\eta_t = -\partial_t \ln(1 - c_t) = \frac{\partial_t c_t}{1 - c_t} \; , \tag{7.2}$$

so that to this order the function $B(g_t, c_t)$ depends also on the second relevant coupling $c_t$. The term $\gamma^{(1)}$ in the square braces of Eq. (7.1) is due to the

interaction-dependence of the mass renormalization $\mu_t$ on the critical trajectory, see the thick solid arrow in Fig.1 and Eq. (5.4). The numerical value of $\gamma^{(1)}$ is given in Eq. (5.8). The second term in the square brace of Eq. (7.1) is due to the momentum-dependent part of the four-point vertex, while the last term is due to the six-point vertex. Hence, to calculate the RG flow at the two-loop order, we have to calculate the anomalous dimension to order $g_t^2$, the momentum-dependent part of the four-point vertex to order $g_t^2$, and the six-point vertex to order $g_t^3$. Note that it would be incorrect to ignore the momentum-dependence of the six-point vertex, because in Eq. (7.1) we need $\Gamma_t^{(6)}(\mathbf{q}, -\mathbf{q}, 0, 0, 0, 0)$ for $|\mathbf{q}| = 1$.

### *7.1 Momentum-dependent part of four-point vertex*

Let us begin with the calculation of the momentum-dependent part of the four-point vertex,

$$\tilde{\Gamma}_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \equiv \Gamma_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) - g_t \ . \tag{7.3}$$

Assuming that the bare four-point vertex is momentum-independent, $\tilde{\Gamma}_t^4(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ satisfies the initial condition at $t = 0$,

$$\tilde{\Gamma}_{t=0}^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = 0 \ . \tag{7.4}$$

Subtracting from the exact flow equation (4.19) for the full four-point vertex the corresponding equation with all external momenta set equal to zero, it is easy to show that to order $g_t^2$ the function $\tilde{\Gamma}_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ satisfies

$$\left[ \partial_t + D - 4 + \sum_{i=1}^{4} \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} \right] \tilde{\Gamma}_t^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = g_t^2 I^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \ , \tag{7.5}$$

where

$$\begin{aligned} I^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = -K_D \Big\langle & C(\hat{\mathbf{q}}' + \mathbf{q}_1 + \mathbf{q}_2) + C(\hat{\mathbf{q}}' + \mathbf{q}_1 + \mathbf{q}_3) \\ & + C(\hat{\mathbf{q}}' + \mathbf{q}_1 + \mathbf{q}_4) - 3C(\hat{\mathbf{q}}') \Big\rangle_{\hat{\mathbf{q}}'} \ . \end{aligned} \tag{7.6}$$

On the right-hand side of Eq. (7.6) we have replaced the exact propagators $G_t(\mathbf{q})$ by the non-interacting propagators without ultraviolet cutoff $C(\mathbf{q})$, see Eq. (6.2). Such a replacement is only justified for $t \gtrsim 1$, so that strictly speaking the approximation (7.6) cannot be used to study the flow of Eq. (7.5) in the interval $0 \leq t \lesssim 1$. However, as discussed by Polchinski [6], the flow

of irrelevant couplings for large $t$ becomes independent of the precise initial conditions (see below), so that for our purpose of investigating the infrared properties of the system we may ignore this subtlety.

To solve Eq. (7.5) with initial condition (7.4), it is convenient to introduce the auxiliary functions

$$\tilde{\Gamma}^{(4)}_{t,s}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \Gamma^{(4)}_t(e^{-s}\mathbf{q}_1, e^{-s}\mathbf{q}_2, e^{-s}\mathbf{q}_3, e^{-s}\mathbf{q}_4) \ , \tag{7.7}$$

$$I^{(4)}_s(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = I^{(4)}(e^{-s}\mathbf{q}_1, e^{-s}\mathbf{q}_2, e^{-s}\mathbf{q}_3, e^{-s}\mathbf{q}_4) \ . \tag{7.8}$$

Then the solution of Eq. (7.5) is given by

$$\tilde{\Gamma}^{(4)}_t(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \tilde{\Gamma}^{(4)}_{t,s=0}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \ , \tag{7.9}$$

where

$$[\partial_t - \partial_s + D - 4]\, \tilde{\Gamma}^{(4)}_{t,s}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = g_t^2 I^{(4)}_s(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \ . \tag{7.10}$$

The initial condition (7.4) implies

$$\tilde{\Gamma}^{(4)}_{t=0,s}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = 0 \ . \tag{7.11}$$

Because in Eq. (7.10) the external momenta $\mathbf{q}_i$ appear simply as parameters, this equation can be considered as a first order partial differential equation with two variables, $t$ and $s$. The solution of Eq. (7.10) with the initial condition (7.11) is easily obtained,

$$\tilde{\Gamma}^{(4)}_{t,s}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \int_0^t d\tau g_\tau^2 e^{-(D-4)(t-\tau)} I^{(4)}_{s+t-\tau}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \ . \tag{7.12}$$

Setting $s = 0$ in Eq. (7.12), we find after a shift of the integration variable

$$\tilde{\Gamma}^{(4)}_t(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \int_0^t d\tau g_{t-\tau}^2 e^{-(D-4)\tau} I^{(4)}_\tau(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \ . \tag{7.13}$$

At the first sight it seems that $\tilde{\Gamma}^{(4)}_t(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ is a non-local function of $g_t$. However, by construction the term $e^{-(D-4)\tau} I^{(4)}_\tau(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ in the integrand vanishes for large $\tau$ at least as fast as $e^{-(D-3)\tau}$, so that the $\tau$- integration is effectively cut off at $\tau \lesssim 1/(D-3)$. For $t \gtrsim 1/(D-3)$ we may then approximate

$g_{t-\tau} \approx g_t$ in the integrand of Eq. (7.13), so that we finally arrive at the behavior anticipated in Eq. (5.11),

$$\tilde{\Gamma}^{(4)}_t(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \sim g_t^2 \gamma^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \quad , \quad t \gtrsim 1/(D-3) \; , \tag{7.14}$$

with

$$\gamma^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \int_0^\infty d\tau e^{-(D-4)\tau} I^{(4)}_\tau(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \; . \tag{7.15}$$

According to Eq. (7.1), for the two-loop flow equation for $g_t$ we need

$$\bar{\gamma}^{(4)} \equiv \left\langle \gamma^{(4)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, 0, 0) \right\rangle_{\hat{\mathbf{q}}} = \int_0^1 d\lambda \lambda^{3-D} \left\langle I^{(4)}(\lambda\hat{\mathbf{q}}, -\lambda\hat{\mathbf{q}}, 0, 0) \right\rangle_{\hat{\mathbf{q}}} \; , \tag{7.16}$$

where we have substituted $\lambda = e^{-\tau}$. Note that the value of $\bar{\gamma}^{(4)}$ is determined by all momentum scales up to the infrared cutoff $\Lambda$. The momentum scale expansion discussed by Morris [13,18] corresponds to expanding the integrand in Eq. (7.16) to some finite order in $\lambda$. Because the upper limit for the $\lambda$-integration is $\lambda = 1$, this expansion is not controlled by a small parameter [13,18]. Obviously, the correct numerical value of the two-loop coefficient of the $\beta$-function can only be obtained if all terms in the momentum scale expansion are resummed [19]. Using the expression for $I^{(4)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ given in Eq. (7.6) and introducing $D$-dimensional spherical coordinates with $\cos\vartheta = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}'$, we find

$$\begin{aligned}\bar{\gamma}^{(4)} = -2K_D \frac{\Omega_{D-1}}{\Omega_D} \int_0^\pi d\vartheta (\sin\vartheta)^{D-2} \int_0^1 d\lambda \lambda^{3-D} \\ \times \left[ \frac{\theta(\lambda + 2\cos\vartheta)}{1 + 2\lambda\cos\vartheta + \lambda^2} - \theta(\cos\vartheta) \right] \; ,\end{aligned} \tag{7.17}$$

where $\Omega_D = (2\pi)^D K_D$ is the volume of the unit sphere in $D$ dimensions. Rearranging terms, $\bar{\gamma}^{(4)}$ can also be written as

$$\begin{aligned}\bar{\gamma}^{(4)} = 2K_D \frac{\Omega_{D-1}}{\Omega_D} \Bigg[ \int_0^{\pi/2} d\vartheta (\sin\vartheta)^{D-2} \int_0^1 d\lambda \frac{\lambda^{4-D}(2\cos\vartheta + \lambda)}{1 + 2\lambda\cos\vartheta + \lambda^2} \\ - \int_0^{\pi/6} d\vartheta (\cos\vartheta)^{D-2} \int_{2\sin\vartheta}^1 d\lambda \frac{\lambda^{3-D}}{1 - 2\lambda\sin\vartheta + \lambda^2} \Bigg] \; ,\end{aligned} \tag{7.18}$$

where in the second integral we have shifted $\vartheta \to \vartheta - \pi/2$. In Sec.7.5 we shall explicitly evaluate this integral for $D = 4$.

### *7.2 Anomalous dimension*

Having determined the momentum-dependent part of the four-point vertex to order $g_t^2$, we can calculate the momentum-dependent part of the two-point vertex to the same order. In analogy with Eq. (7.3), let us define

$$\tilde{\Gamma}_t^{(2)}(q) \equiv \Gamma_t^{(2)}(\mathbf{q}, -\mathbf{q}) - \mu_t \ . \tag{7.19}$$

Note that according to Eq. (5.10) we expect for large $t$,

$$\tilde{\Gamma}_t^{(2)}(q) \sim c_t q^2 + g_t^2 \gamma^{(2)}(q) + O(g_t^3) \ . \tag{7.20}$$

Hence, the flow of our second relevant coupling $c_t$ (which according to Eq. (7.2) determines the anomalous dimension $\eta_t$) can be obtained from the quadratic term in the expansion of $\tilde{\Gamma}_t^{(2)}(q)$ for small $q$. The function $\gamma^{(2)}(q)$, which by construction vanishes faster than $q^2$ for small $q$, is not needed for a two-loop calculation.

Subtracting from the exact flow equation (4.8) for the irreducible two-point vertex the same equation with $\mathbf{q}$ set equal to zero, we obtain to leading order in $g_t$

$$\left[\partial_t - 2 + \mathbf{q} \cdot \nabla_{\mathbf{q}}\right] \tilde{\Gamma}_t^{(2)}(q) = g_t^2 I^{(2)}(q) + O(g_t^3) \ , \tag{7.21}$$

where

$$I^{(2)}(q) = \frac{K_D}{2} \left\langle \gamma^{(4)}(\mathbf{q}, -\mathbf{q}, \hat{\mathbf{q}}', -\hat{\mathbf{q}}') - \gamma^{(4)}(0, 0, \hat{\mathbf{q}}', -\hat{\mathbf{q}}') \right\rangle_{\hat{\mathbf{q}}'} \ . \tag{7.22}$$

Expanding both sides of Eq. (7.21) in powers of $q$, we obtain to leading order

$$\partial_t c_t = \alpha_2 g_t^2 + O(g_t^3) \ , \tag{7.23}$$

where

$$\alpha_2 = \left. \frac{\partial I^{(2)}(q)}{\partial (q^2)} \right|_{q^2 = 0} \ . \tag{7.24}$$

From Eq. (7.23) we conclude that the weak coupling expansion of the function $A(g_t, c_t)$ defined in Eq. (5.9) is

$$A(g_t, c_t) = \alpha_2 g_t^2 + O(g_t^3, g_t c_t) \ . \tag{7.25}$$

Eq. (7.23) is easily integrated,

$$c_t = \alpha_2 \int_0^t dt' g_{t'}^2 \ . \tag{7.26}$$

Hence, the relation between $c_t$ and $g_t$ is non-local, so that in general also the anomalous dimension $\eta_t$ given in Eq. (7.2) is a non-local function of $g_t$. This is due to the fact that $c_t$ is not irrelevant. However, to leading order the term in the denominator of Eq. (7.2) can be neglected, so that

$$\eta_t = \partial_t c_t + O(g_t^4) = \alpha_2 g_t^2 + O(g_t^3) \ , \tag{7.27}$$

i.e. to this order $\eta_t$ is a local function of $g_t$.

For an explicit calculation of the number $\alpha_2$, we need to calculate the leading $q$-dependence of the function $I^{(2)}(q)$. Using Eqs.(7.22), (7.15) and (7.6), we find

$$I^{(2)}(q) = -K_D^2 \int_0^1 d\lambda \lambda^{3-D} \left\langle f(\lambda|\hat{\mathbf{q}}' + \mathbf{q}|) - f(\lambda) \right\rangle_{\hat{\mathbf{q}}'} \ , \tag{7.28}$$

where we have defined

$$f(\lambda) = \left\langle \frac{\theta(|\hat{\mathbf{q}}'' + \lambda \hat{\mathbf{e}}|) - 1}{(\hat{\mathbf{q}}'' + \lambda \hat{\mathbf{e}})^2} \right\rangle_{\hat{\mathbf{q}}''} = \frac{\Omega_{D-1}}{\Omega_D} \int_{-\lambda/2}^{1} dx \frac{(1-x^2)^{\frac{D-3}{2}}}{1 + 2\lambda x + \lambda^2} \ . \tag{7.29}$$

Here $\hat{\mathbf{e}}$ is an arbitrary constant unit vector. Apart from a different normalization, our function $f(\lambda)$ agrees with the corresponding function introduced in Appendix D of Ref. [11], were the anomalous dimension at the Wilson-Fisher fixed point has been calculated to order $(4-D)^2$. Expanding $I^{(2)}(q) = \alpha_2 q^2 + O(q^3)$ we find from Eq. (7.28)

$$\alpha_2 = -\frac{K_D^2}{2D} \left[ (D-1) \int_0^1 d\lambda \lambda^{4-D} f'(\lambda) + \int_0^1 d\lambda \lambda^{5-D} f''(\lambda) \right] \ . \tag{7.30}$$

In Sec.7.5 we shall explicitly evaluate Eq. (7.30) in four dimensions.

To complete the two-loop calculation of the function $B(g_t, c_t)$ given in Eq. (7.1), we need to know the number

$$\bar{\gamma}^{(6)} = \left\langle \gamma^{(6)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, 0, 0, 0, 0) \right\rangle_{\hat{\mathbf{q}}} . \tag{7.31}$$

Recall that according to Eq. (5.12) the dimensionless function $\gamma^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6)$ is defined in terms of the large $t$-behavior of the irreducible six-point vertex,

$$\Gamma_t^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) \sim g_t^3 \gamma^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_n) + O(g_t^4) . \tag{7.32}$$

From the exact flow-equation (4.20) we find that to order $g_t^3$ the irreducible six-point vertex satisfies

$$\left[ \partial_t + 2D - 6 + \sum_{i=1}^{6} \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} \right] \Gamma_t^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) = g_t^3 I^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) , \tag{7.33}$$

where

$$I^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) = K_D \sum_{\{I_1, I_2\}, I_3}^{45 \text{ terms}} \left\langle C(\hat{\mathbf{q}}' + \mathbf{Q}_1) C(\hat{\mathbf{q}}' - \mathbf{Q}_2) \right\rangle_{\hat{\mathbf{q}}'} . \tag{7.34}$$

Eq. (7.33) can be solved with the same method as Eq. (7.5). The solution with initial condition $\Gamma_{t=0}^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) = 0$ is

$$\Gamma_t^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) = \int_0^t d\tau g_{t-\tau}^3 e^{-(2D-6)\tau} I^{(6)}(e^{-\tau}\mathbf{q}_1, \ldots, e^{-\tau}\mathbf{q}_6) , \tag{7.35}$$

which is the analog of Eq. (7.13). For $t \gtrsim 1/(2D-6)$ we may approximate $g_{t-\tau} \approx g_t$ under the integral, so that Eq. (7.35) indeed reduces to Eq. (5.12), with

$$\gamma^{(6)}(\mathbf{q}_1, \ldots, \mathbf{q}_6) = \int_0^1 d\lambda \lambda^{2D-7} I^{(6)}(\lambda\mathbf{q}_1, \ldots, \lambda\mathbf{q}_6) , \tag{7.36}$$

where we have substituted $\lambda = e^{-\tau}$. For our two-loop calculation we only need $\gamma^{(6)}(\hat{\mathbf{q}}, -\hat{\mathbf{q}}, 0, 0, 0, 0)$. Using Eq. (7.34) this can be written as

$$\gamma^{(6)}(\hat{\mathbf{q}},-\hat{\mathbf{q}},0,0,0,0)=K_D\int_0^1 d\lambda\lambda^{2D-7}\left[9\left\langle C^2(\hat{\mathbf{q}}')\right\rangle_{\hat{\mathbf{q}}'}\right.$$

$$\left.+\,12\left\langle C(\hat{\mathbf{q}}'-\lambda\hat{\mathbf{q}})C(\hat{\mathbf{q}}')+C(\hat{\mathbf{q}}'+\lambda\hat{\mathbf{q}})C(\hat{\mathbf{q}}')+C^2(\hat{\mathbf{q}}'+\lambda\hat{\mathbf{q}})\right\rangle_{\hat{\mathbf{q}}'}\right]\ . \qquad (7.37)$$

For the evaluation of $\langle C^2(\hat{\mathbf{q}}')\rangle_{\hat{\mathbf{q}}'}$ one should be careful to properly define the value of the step function $\theta(x)$ at $x=0$. Following the procedure discussed by Morris [13], we find

$$\left\langle C^2(\hat{\mathbf{q}}')\right\rangle_{\hat{\mathbf{q}}'}=\frac{1}{3}\ . \qquad (7.38)$$

To obtain the number $\bar{\gamma}^{(6)}$ defined in Eq. (7.31), we should average Eq. (7.37) over all directions of the unit vector $\hat{\mathbf{q}}$. Using $D$-dimensional spherical coordinates we obtain

$$\bar{\gamma}^{(6)}=\frac{3K_D}{2(D-3)}+12K_D\frac{\Omega_{D-1}}{\Omega_D}\int_0^\pi d\vartheta(\sin\vartheta)^{D-2}\int_0^1 d\lambda\lambda^{2D-7}$$
$$\times\,\Theta(\lambda+2\cos\vartheta)\left[\frac{1}{1+2\lambda\cos\vartheta+\lambda^2}+\frac{1}{(1+2\lambda\cos\vartheta+\lambda^2)^2}\right]\ . \qquad (7.39)$$

For an explicit evaluation of Eq. (7.39) in $D=4$ see Sec.7.5.

### *7.4 $\beta$-function*

Collecting all contributions to the function $B(g_t,c_t)$ given in Eq. (7.1) and using the fact that the dependence on $c_t$ enters only via the anomalous dimension $\eta_t=\alpha_2 g_t^2+O(g_t^3)$, we see that at the two-loop order the flow equation (5.9) for $g_t$ can be written as

$$\partial_t g_t=B(g_t,c_t)=\beta_0 g_t+\beta_1 g_t^2+\beta_2 g_t^3+O(g_t^4)\ , \qquad (7.40)$$

with

$$\beta_0=4-D\ \ ,\ \ \beta_1=-\frac{3K_D}{2}\ \ ,\ \ \beta_2=\beta_2^\mu+\beta_2^\eta+\beta_2^{(4)}+\beta_2^{(6)}\ , \qquad (7.41)$$

where

$$\beta_2^\mu=3K_D\gamma^{(1)}=-\frac{3K_D^2}{2(D-2)}\ , \qquad (7.42)$$

$$\beta_2^{\eta} = -2\alpha_2 \ , \tag{7.43}$$

$$\beta_2^{(4)} = -3K_D\bar{\gamma}^{(4)} \ , \tag{7.44}$$

$$\beta_2^{(6)} = \frac{K_D}{2}\bar{\gamma}^{(6)} \ . \tag{7.45}$$

The term $\beta_2^{\mu}$ arises from flow of the momentum-independent part of the two-point vertex along the critical surface, see Eq. (5.8). The second term, $\beta_2^{\eta}$, is due to the flow of the anomalous dimension, $\beta_2^{(4)}$ is due to the momentum-dependent part of the four-point vertex, and $\beta_2^{(6)}$ is due to the six-point vertex. Note that the numerical values of $\beta_2^{(4)}$ and $\beta_2^{(6)}$ are determined by the behavior of the four-point and six-point vertices at finite momenta, and thus contain the effect of an infinite number of irrelevant couplings.

### *7.5 Four dimensions*

We shall now explicitly evaluate the two-loop coefficient $\beta_2$ in $D = 4$, and show that it agrees with the result obtained within the field theory approach [1].

According to Eq. (7.42), the term $\beta_2^{\mu}$ is in $D = 4$ given by

$$\beta_2^{\mu} = -\frac{3K_4^2}{4} \quad , \quad K_4 = \frac{2}{(4\pi)^2} \ . \tag{7.46}$$

Next, consider the contribution $\beta_2^{\eta}$ due to the anomalous dimension. After an integration by parts we obtain from Eq. (7.30) in $D = 4$,

$$\alpha_2 = \frac{K_4^2}{8}\left[2f(1) - 2f(0) + f'(1)\right] \ . \tag{7.47}$$

The required numbers are (see also Appendix D of Ref.[11])

$$f(0) = \frac{1}{2} \quad , \quad f(1) = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} \quad , \quad f'(1) = -\frac{2}{3} + \frac{\sqrt{3}}{\pi} \ , \tag{7.48}$$

so that

$$\alpha_2 = \frac{K_4^2}{24} \ . \tag{7.49}$$

Thus, the flowing anomalous dimension in $D = 4$ is

$$\eta_t = \frac{K_4^2}{24}g_t^2 + O(g_t^3) \ , \tag{7.50}$$

in agreement with the corresponding result of the field theory method [1]. From Eq. (7.43) we conclude that the contribution to the anomalous dimension of the field to the two-loop coefficient of the $\beta$-function is

$$\beta_2^\eta = -\frac{K_4^2}{12} \; . \tag{7.51}$$

The calculation of $\beta_2^{(4)} = -3K_D\bar{\gamma}^{(4)}$ is quite tedious. We first perform the $\lambda$-integrations in the expression for $\bar{\gamma}^{(4)}$ given in Eq. (7.18). In $D = 4$ the integrals can be evaluated analytically,

$$\int\limits_0^1 d\lambda \frac{2\cos\vartheta + \lambda}{1 + 2\lambda\cos\vartheta + \lambda^2} = \frac{\vartheta}{2}\cot\vartheta + \frac{1}{2}\ln[2(1+\cos\vartheta)] \; , \tag{7.52}$$

$$\int\limits_{2\sin\vartheta}^1 d\lambda \frac{1}{\lambda(1 - 2\lambda\sin\vartheta + \lambda^2)} = \left(\frac{\pi}{4} - \frac{3\vartheta}{2}\right)\tan\vartheta - \frac{1}{2}\ln[8(1-\sin\vartheta)] - \ln\sin\vartheta \; . \tag{7.53}$$

Substituting Eqs.(7.52) and (7.53) into Eq. (7.18), most of the $\vartheta$-integrations can also be performed analytically. After several integrations by parts the final result for $\beta_2^{(4)}$ can be cast into the following form

$$\beta_2^{(4)} = K_4^2\left[-\frac{1}{2} + \frac{3\sqrt{3}}{2\pi} - 2\ln 2 + \frac{6}{\pi}L\left(\frac{\pi}{3}\right)\right] \; , \tag{7.54}$$

where

$$L(x) = -\int\limits_0^x d\vartheta \ln\cos\vartheta \tag{7.55}$$

is known as Lobachevskiy's function [25]. With $L(\frac{\pi}{3}) = 0.218391\ldots$ the numerical value of $\beta_2^{(4)}$ turns out to be

$$\beta_2^{(4)} \approx -K_4^2 \times 0.642204\ldots \; . \tag{7.56}$$

Finally, let us evaluate the contribution $\beta_2^{(6)} = \frac{K_D}{2}\bar{\gamma}^{(6)}$ from the six-point vertex to the two-loop coefficient of the $\beta$-function in $D = 4$. To calculate the

integral $\bar{\gamma}^{(6)}$ in Eq. (7.39), it is convenient to first integrate by parts. Writing

$$\frac{1}{(1+2\lambda\cos\vartheta+\lambda^2)^2} = \frac{1}{2\lambda\sin\vartheta}\frac{\partial}{\partial\vartheta}\frac{1}{1+2\lambda\cos\vartheta+\lambda^2} \ , \tag{7.57}$$

we obtain after some rearrangements

$$\begin{aligned}\beta_2^{(6)} = K_4^2\Bigg\{ &\frac{7}{4} + \frac{3\sqrt{3}}{2\pi} + \frac{6}{\pi}\int\limits_0^{\pi/2} d\vartheta \int\limits_0^1 d\lambda \frac{2\lambda\sin^2\vartheta - \cos\vartheta}{1+2\lambda\cos\vartheta+\lambda^2} \\ &+ \frac{6}{\pi}\int\limits_0^{\pi/6} d\vartheta \int\limits_{2\sin\vartheta}^1 d\lambda \frac{2\lambda\cos^2\vartheta + \sin\vartheta}{1-2\lambda\sin\vartheta+\lambda^2}\Bigg\} \ .\end{aligned} \tag{7.58}$$

The $\lambda$-integrations can now be performed analytically,

$$\int\limits_0^1 d\lambda \frac{2\lambda\sin^2\vartheta - \cos\vartheta}{1+2\lambda\cos\vartheta+\lambda^2} = -\frac{\vartheta}{2}\left[\cot\vartheta + \sin(2\vartheta)\right] + \sin^2\vartheta \ln[2(1+\cos\vartheta)] \ , \tag{7.59}$$

$$\begin{aligned}&\int\limits_{2\sin\vartheta}^1 d\lambda \frac{2\lambda\cos^2\vartheta + \sin\vartheta}{1-2\lambda\sin\vartheta+\lambda^2} = \\ &\qquad \left(\frac{\pi}{4} - \frac{3\vartheta}{2}\right)\left[\tan\vartheta + \sin(2\vartheta)\right] + \cos^2\vartheta \ln[2(1-\sin\vartheta)] \ .\end{aligned} \tag{7.60}$$

Substituting Eqs.(7.59) and (7.60) into Eq. (7.58) and performing the $\vartheta$-integrations we obtain after some tedious re-shufflings

$$\beta_2^{(6)} = K_4^2\left[\frac{11}{4} - \frac{3\sqrt{3}}{2\pi} + 2\ln 2 - \frac{6}{\pi}L\left(\frac{\pi}{3}\right)\right] \ . \tag{7.61}$$

In deriving this expression, we have used the fact that for $0 \leq x < \frac{\pi}{4}$ Lobachevskiy's function satisfies the functional relationship [25]

$$L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right)\ln 2 - \frac{1}{2}L\left(\frac{\pi}{2} - 2x\right) \ , \tag{7.62}$$

which implies in particular

$$\frac{1}{2}\ln 2 - \frac{6}{\pi}L\left(\frac{\pi}{3}\right) + \frac{9}{\pi}L\left(\frac{\pi}{6}\right) = 0 \ . \tag{7.63}$$

We conclude that the contribution of the six-point vertex to the two-loop coefficient of the $\beta$-function has the numerical value

$$\beta_2^{(6)} = K_4^2 \times 2.892204\ldots \,. \tag{7.64}$$

Neither $\beta_2^{(4)}/K_4^2$ nor $\beta_2^{(6)}/K_4^2$ is a rational number, but their sum is: adding Eqs.(7.54) and (7.61), we find

$$\beta_2^{(4)} + \beta_2^{(6)} = \frac{9}{4} K_4^2 \,. \tag{7.65}$$

Collecting all terms, we obtain for the two-loop coefficient of the RG $\beta$-function

$$\beta_2 = \beta_2^{\mu} + \beta_2^{\eta} + \beta_2^{(4)} + \beta_2^{(6)} = \left[ -\frac{3}{4} - \frac{1}{12} + \frac{9}{4} \right] K_4^2 = \frac{17}{12} K_4^2 \,, \tag{7.66}$$

which agrees with the result obtained by means of the field theory approach [1]. The two-loop RG-flow in the plane of the two relevant couplings $g_t$ and $c_t$ in $D = 4$ is shown in Fig.2.

## 8 Conclusions

In this work we have shown how the known two-loop results for the RG $\beta$-function of a simple massless scalar field theory can be obtained from the exact Wilsonian RG. Although from a technical point of view the two-loop calculation within the exact RG is more tedious than the same calculation within the orthodox field theory method [1], the Wilsonian RG is conceptually simpler, because there is no need to invoke rather abstract concepts such as dimensional regularization or minimal subtraction. Moreover, the exact RG has the advantage that it can be used even if the problem of interest cannot be mapped onto a renormalizable field theory. This is of particular importance in condensed matter systems, where an underlying lattice spacing always furnishes a physical ultraviolet cutoff, and there is no need for taking the continuum limit. An interesting system where the orthodox field theory approach fails is a two-dimensional electronic system with a Fermi surface that contains saddle points [26]. These give rise to van Hove singularities in the electronic density of states, which in turn generate non-renormalizable singularities in perturbation theory. As a consequence, this problem cannot be mapped onto a renormalizable field theory [27].

Due to its generality and conceptual simplicity, the exact Wilsonian RG is very popular in condensed matter theory and statistical physics. However,

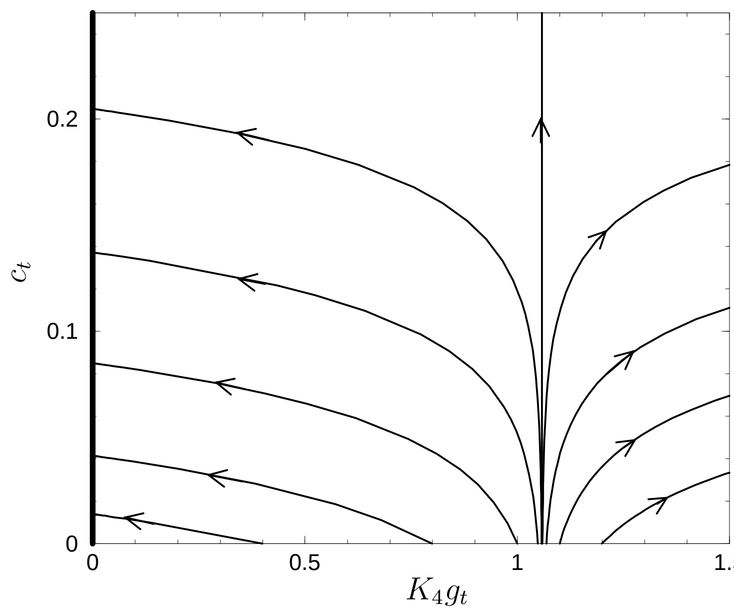


Fig. 2. Two-loop RG-flow of the massless theory for $D = 4$ in the $g$-$c$ plane. The flow is determined by $\partial_t c_t = \frac{1}{24}\tilde{g}_t^2$ and $\partial_t \tilde{g}_t = -\frac{3}{2}\tilde{g}_t^2 + \frac{17}{12}\tilde{g}_t^3$, where $\tilde{g}_t = K_4 g_t$. The thick line represents the Gaussian fixed point manifold, which can be parameterized by the value of the wave-function renormalization $Z_{t=\infty} = 1 - c_{t=\infty}$, or alternatively, the initial interaction $g_{t=0}$. Note that in $D = 4$ the two-loop $\beta$-function (7.40) vanishes at $K_4 g^* = \frac{18}{17} \approx 1.058823\ldots$. However, this is not a fixed point of the RG, because for $g_t \geq g^*$ the second relevant coupling $c_t$ flows to strong coupling. Because this graph is based on an expansion of the RG flow equations in powers of $g_t$ and $c_t$, only the weak coupling regime $K_4 g_t \ll 1$ and $c_t \ll 1$ can be trusted.

even condensed matter theorists use the field theory method for performing two-loop RG calculations [9]. In the last few years the exact RG has been used to study the fermionic many-body problem [28]. To the best of our knowledge, two-loop RG calculations have not been performed for fermionic systems in $D > 1$. But also for one-dimensional fermionic systems, where the conventional field theory version of the RG method has been applied for many years [29], an ambiguity inherent in the conventional way of performing two-loop calculations has recently been pointed out [30]. It seems to us that the exact Wilsonian RG, in its irreducible form used in this work, is the most promising method for performing two-loop calculations for fermionic condensed matter systems. This formulation of the exact RG can be used even if the continuum limit does not exist, and has the potential of clarifying ambiguities of the type discussed in Ref. [30].

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**Appendix: Four types of generating functionals**

Here we briefly summarize some basic definitions and representations of generating functionals for various types of correlation functions. Although most of the following equations can be found in textbooks [1], we believe that this Appendix greatly enhances the readability of this work by giving a concise summary of the relevant expressions.

*A.1 Disconnected Green functions*

The generating functional of the disconnected Green functions is given by the normalized partition function in the presence of external sources,

$$\mathcal{G}\{J\} = \frac{\mathcal{Z}\{J\}}{\mathcal{Z}\{0\}} = \frac{\int \mathcal{D}\{\phi\} e^{-S\{\phi\}+(J,\phi)}}{\int \mathcal{D}\{\phi\} e^{-S\{\phi\}}} \; , \tag{A.1}$$

where we have used the notation

$$(J, \phi) = \int d\mathbf{r} J(\mathbf{r})\phi(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^D} J_{\mathbf{k}} \phi_{-\mathbf{k}} \; . \tag{A.2}$$

The disconnected $n$-point functions are defined by

$$G^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) \equiv \langle \phi(\mathbf{r}_1) \ldots \phi(\mathbf{r}_n) \rangle = \left. \frac{\delta^{(n)} \mathcal{G}\{J\}}{\delta J(\mathbf{r}_1) \ldots \delta J(\mathbf{r}_n)} \right|_{J=0} \; . \tag{A.3}$$

## A.2 Connected Green functions

The generating functional $\mathcal{G}_{\rm c}\{J\}$ of the connected Green functions can be defined by

$$e^{\mathcal{G}_{\rm c}\{J\}} = \frac{\mathcal{Z}\{J\}}{\mathcal{Z}_0} = \frac{\int \mathcal{D}\{\phi\} e^{-S\{\phi\}+(J,\phi)}}{\int \mathcal{D}\{\phi\} e^{-S_0\{\phi\}}} \ . \tag{A.4}$$

Note that by definition $-\mathcal{G}_{\rm c}\{J=0\} = F - F_0$, where $F$ is the free energy of the interacting system, and $F_0 = -\ln \mathcal{Z}_0$ is the free energy of the non-interacting one. The connected $n$-point functions can then be written as

$$G_{\rm c}^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) \equiv \langle \phi(\mathbf{r}_1) \ldots \phi(\mathbf{r}_n) \rangle_{\rm c} = \left. \frac{\delta^{(n)} \mathcal{G}_{\rm c}\{J\}}{\delta J(\mathbf{r}_1) \ldots \delta J(\mathbf{r}_n)} \right|_{J=0} \ . \tag{A.5}$$

In contrast to Eq. (A.1), we have normalized the integral in Eq. (A.4) with the non-interacting partition function, such that $-\mathcal{G}_{\rm c}\{0\}$ is just the change of the free energy due to the interaction. This normalization is useful, because then $\mathcal{G}_{\rm c}\{J\}$ can be written in terms of functional differential operators as follows,

$$\begin{aligned} e^{\mathcal{G}_{\rm c}\{J\}} &= e^{-S_{\rm int}\{\frac{\delta}{\delta J}\}} e^{\frac{1}{2}(J, G_0 J)} \\ &= e^{\frac{1}{2}(J, G_0 J)} \left[ e^{\frac{1}{2}(\frac{\delta}{\delta \phi}, G_0 \frac{\delta}{\delta \phi})} e^{-S_{\rm int}\{\phi\}} \right]_{\phi = G_0 J} \ . \end{aligned} \tag{A.6}$$

## A.3 Amputated connected Green functions

The generating functional of the amputated connected Green functions is

$$\mathcal{G}_{\rm ac}\{\phi\} = \left[ \mathcal{G}_{\rm c}\{J\} - \frac{1}{2}(J, G_0 J) \right]_{J = G_0^{-1}\phi} = \mathcal{G}_{\rm c}\{G_0^{-1}\phi\} - \frac{1}{2}(\phi, G_0^{-1}\phi) \ . \tag{A.7}$$

The amputated connected Green functions are then

$$G_{\rm ac}^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \left. \frac{\delta^{(n)} \mathcal{G}_{\rm ac}\{\phi\}}{\delta \phi(\mathbf{r}_1) \ldots \delta \phi(\mathbf{r}_n)} \right|_{\phi=0} \ . \tag{A.8}$$

From the definition (A.7) and Eqs.(A.4) and (A.1), it is easy to show that $\mathcal{G}_{\rm ac}\{\phi\}$ has the following functional integral representation,

$$e^{\mathcal{G}_{\rm ac}\{\phi\}} = \frac{\int \mathcal{D}\{\phi'\} e^{-S_0\{\phi'\}-S_{\rm int}\{\phi'+\phi\}}}{\int \mathcal{D}\{\phi'\} e^{-S_0\{\phi'\}}} \, . \tag{A.9}$$

Furthermore, from the definition (A.7) and the second line of Eq. (A.6) we see that $\mathcal{G}_{\rm ac}\{\phi\}$ has the following representation in terms of a functional differential operator,

$$e^{\mathcal{G}_{\rm ac}\{\phi\}} = e^{\frac{1}{2}(\frac{\delta}{\delta\phi}, G_0 \frac{\delta}{\delta\phi})} e^{-S_{\rm int}\{\phi\}} \, . \tag{A.10}$$

### *A.4 Irreducible Green function*

The generating functional of the one-particle irreducible Green functions contains the information about the many-body correlations in the most compact form. This generating functional is obtained from the generating functional of the connected Green functions via a Legendre transformation,

$$\mathcal{L}\{\varphi\} = (\varphi, J) - \mathcal{G}_{\rm c}\{J\{\varphi\}\} \, , \tag{A.11}$$

where $J\{\varphi\}$ is defined as a functional of the classical field $\varphi$ via

$$\varphi(\mathbf{r}) = \langle \phi(\mathbf{r}) \rangle_{\rm c} = \frac{\delta \mathcal{G}_{\rm c}\{J\}}{\delta J(\mathbf{r})} \, . \tag{A.12}$$

Hence,

$$\mathcal{L}\{\varphi\} = F - F_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\mathbf{r}_1 \ldots \int d\mathbf{r}_n L^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) \varphi(\mathbf{r}_1) \ldots \varphi(\mathbf{r}_n) \, , \tag{A.13}$$

where

$$L^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \left. \frac{\delta^{(n)} \mathcal{L}\{\varphi\}}{\delta\varphi(\mathbf{r}_1) \ldots \delta\varphi(\mathbf{r}_n)} \right|_{\varphi=0} \, . \tag{A.14}$$

Note that in the absence of interactions ($S_{\rm int} = 0$) the only non-zero vertex is

$$L_0^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = G_0^{-1}(\mathbf{r}_1 - \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \left[ -\nabla^2_{\mathbf{r}_2} + m_0^2 \right] \, . \tag{A.15}$$

For the interacting system we have by definition

$$L^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = G_0^{-1}(\mathbf{r}_1 - \mathbf{r}_2) + \Sigma(\mathbf{r}_1 - \mathbf{r}_2) \ , \tag{A.16}$$

where $\Sigma(\mathbf{r})$ is the irreducible self-energy. To obtain the generating functional of all irreducible $n$-point functions (including the two-point function), we should therefore subtract the free propagator from Eq. (A.11), defining

$$\mathcal{G}_{\rm ir}\{\varphi\} = \mathcal{L}\{\varphi\} - \frac{1}{2}(\varphi, G_0^{-1}\varphi) = (\varphi, J) - \frac{1}{2}(\varphi, G_0^{-1}\varphi) - \mathcal{G}_{\rm c}\{J\{\varphi\}\} \ . \tag{A.17}$$

The irreducible $n$-point functions are then

$$G_{\rm ir}^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \left. \frac{\delta^{(n)}\mathcal{G}_{\rm ir}\{\varphi\}}{\delta\varphi(\mathbf{r}_1)\ldots\delta\varphi(\mathbf{r}_n)} \right|_{\varphi=0} \ . \tag{A.18}$$

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